Successive and Partial Differentiation

5

5.1. SUCCESSIVE DIFFERENTIATION

If y is a function of x, its derivative $\frac{dy}{dx}$ is itself a function of x. In general, we assume that it also possesses a derivative if it is differentiated further. The derivative $\frac{dy}{dx}$ is called the first differential co-efficient or first derivative of y w.r.t. x. The differential co-efficient of $\frac{dy}{dx}$, *i.e.*, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second differential co-efficient or **second derivative** of y w.r.t. x, which we denote as $\frac{d^2y}{dx^2}$ [read as *dee two y* over *dee x square*]. In general, the n^{th} differential co-efficient of y w.r.t. x is denoted by $\frac{d^ny}{dx^n}$. The process of finding the differential co-efficient of a function again and again is called **Successive Differentiation.**

Thus, if $y = f(x)$, the successive differential co-efficients of $f(x)$ are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots, \frac{d^ny}{dx^n}.$$

These are also denoted by :

(*i*) $y_1, y_2, y_3, \ldots\ldots, y_n$.

(*ii*) $y', y'', y''', \ldots\ldots, y^{(n)}$.

(*iii*) $Dy, D^2y, D^3y, \ldots\ldots, D^ny$.

(*iv*) $f'(x), f''(x), f'''(x), \ldots\ldots, f^n(x)$.

5.2. CALCULATION OF n^{th} ORDER DERIVATIVES

I. The n^{th} differential co-efficient of x^m.

Let $\quad y = x^m$

then $\quad y_1 = mx^{m-1}$

$$y_2 = m(m-1)\, x^{m-2}$$

$$y_3 = m(m-1)(m-2)\, x^{m-2}$$

....

....

$\therefore \qquad y_n = [m(m-1)(m-2) \text{...... upto } n \text{ factors}] \times x^{m-n}$

$= m(m-1)(m-2) \text{......} (m-n+1) \,.\, x^{m-n}$, where $n < m$.

[**Cor.** If m be a positive integer, and if $n = m$

then $\qquad y_m = m(m-1)(m-2) \text{......} (m-m+1)\, x^{m-m}$

$= m(m-1)(m-2) \text{......} 3.2.1. = m\,!$

i.e., $$\boxed{\frac{d^m}{dx^m}(x^m) = m\,!}$$

and $(m+1)^{th}$, $(m+2)^{th}$ derivative ...... etc., each will be = 0.

$\therefore$ $y_m = m\,!$ which is constant and $y_{(m+1)}$ which is the derivative of y_m will be zero and so on.]

II. The n^{th} differential co-efficient of $(ax+b)^m$, where $n < m$.

Let $\qquad y = (ax+b)^m$

then $\qquad y_1 = m(ax+b)^{m-1} \,.\, a$

$y_2 = m(m-1)(ax+b)^{m-2} \,.\, a^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} \,.\, a^3$

....

....

$$\therefore \qquad \boxed{y_n = m(m-1)(m-2)(m-3) \text{......} (m-n+1)(ax+b)^{m-n} \,.\, a^n}$$

[**Cor.** If $n = m$, then

$$y_m = m(m-1)(m-2) \text{......} 3.2.1. (ax+b)^{\circ}\, a^m = \underline{|m}\, a^m]$$

III. The n^{th} differential co-efficient of $\frac{1}{ax+b}\left[x \neq -\frac{b}{a}\right]$

Let $\qquad y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

then $\qquad y_1 = (-1)(ax+b)^{-2} \,.\, a$

$y_2 = (-1)(-2)(ax+b)^{-3} \,.\, a^2 = (-1)^2 \,.\, 2\,!\,(ax+b)^{-3} \,.\, a^2$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \,.\, a^3 = (-1)^3 \,.\, 3\,!\,(ax+b)^{-4} \,.\, a^3$

....

....

$\therefore \qquad y_n = (-1)^n \,.\, n\,!\,(ax+b)^{-(n+1)} \,.\, a^n$

or $$\boxed{y_n = \frac{(-1)^n\, n\,!\, a^n}{(ax+b)^{n+1}}.}$$

IV. The n^{th} differential co-efficient of log $(ax+b)$.

Let $\qquad y = \log(ax+b)$

then $\qquad y_1 = \dfrac{1}{ax+b}\, a = (ax+b)^{-1} \,.\, a$

$$y_2 = (-1)(ax+b)^{-2} \cdot a^2$$
$$y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3 = (-1)^2\, 2\,!\,(ax+b)^{-3} \cdot a^3$$

....

....

$\therefore$ $$y_n = (-1)^{n-1}\,(n-1)\,!\,(ax+b)^{-n} \cdot a^n$$

or $$\boxed{\mathbf{y_n = \frac{(-1)^{n-1}\,(n-1)\,!\,a^n}{(ax+b)^n}}}$$

$\left[\textbf{Cor.}\text{ If } a = 1;\, b = 0,\, y = \log x, \text{ then } y_n = \dfrac{(-1)^{n-1}\,(n-1)\,!}{x^n}\right]$

V. The n^{th} differential co-efficient of a^{mx}

Let $$y = a^{mx}$$

$\therefore$ $$y_1 = m \cdot a^{mx}\,(\log a)$$
$$y_2 = m^2 \cdot a^{mx}\,(\log a)^2$$
$$y_3 = m^3 \cdot a^{mx}\,(\log a)^3$$

....

....

$\therefore$ $$\boxed{\mathbf{y_n = m^n \cdot a^{mx}\,(\log a)^n}}$$

[**Cor.** Put $a = e$, then $y = \mathbf{e^{mx}}$ and $\mathbf{y_n = m^n\, e^{mx}}$.]

VI. The n^{th} differential co-efficient of sin (ax + b)

Let $$y = \sin(ax+b)$$

then $$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right) \qquad \text{[Note this step]}$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$= a^2\sin\left(ax+b+2\cdot\frac{\pi}{2}\right)$$
$$y_3 = a^3\cos\left(ax+b+2\cdot\frac{\pi}{2}\right) = a^3\sin\left(ax+b+2\cdot\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$= a^3\sin\left(ax+b+3\cdot\frac{\pi}{2}\right)$$

....

....

$$\mathbf{y_n = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)}$$

$\left[\textbf{Cor.}\text{ If } b = 0,\, y = \sin ax,\, y_n = a^n \sin\left(ax + n\dfrac{\pi}{2}\right)\right]$

VII. The n^{th} differential co-efficient of cos (ax + b)

Let $$y = \cos(ax+b)$$

then $$y_1 = -\sin(ax+b) \cdot a = a\cos\left(ax+b+\frac{\pi}{2}\right) \qquad \text{[Note this step]}$$

$$y_2 = -a^2 \sin\left(ax + b + \frac{\pi}{2}\right) = a^2 \cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= a^2 \cos\left(ax + b + 2 \cdot \frac{\pi}{2}\right)$$
$$y_3 = -a^3 \sin\left(ax + b + 2 \cdot \frac{\pi}{2}\right) = a^3 \cos\left(ax + b + 2 \cdot \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= a^3 \cos\left(ax + b + 3 \cdot \frac{\pi}{2}\right)$$

....

....

$$\boxed{\mathbf{y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right)}}$$

$$\left[\textbf{Cor. } \text{If } y = \cos ax,\ y_n = a^n \cos\left(ax + n \cdot \frac{\pi}{2}\right).\right]$$

VIII. The n^{th} differential co-efficient of $e^{ax} \sin(bx + c)$

Let $\quad y = e^{ax} \sin(bx + c)$

then $\quad y_1 = e^{ax} \cdot \cos(bx + c) \cdot b + \sin(bx + c) \cdot a \cdot e^{ax}$

$$= e^{ax} [a \sin(bx + c) + b \cos(bx + c)].$$

We determine two constants r and θ, to change the expression into a single sine which will enable us to make the required generalisation by putting, $a = r \cos\theta$, $b = r \sin\theta$

$\therefore \quad r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1} \dfrac{b}{a}$.

Hence $\quad y_1 = e^{ax} [r \cos\theta \cdot \sin(bx + c) + r \sin\theta \cdot \cos(bx + c)]$

$$= r\, e^{ax} \sin(bx + c + \theta).$$

Thus y_1 is obtained from y on multiplying it by the constant r and increasing the angle by the constant θ. Repeating the same rule to y_1, we have

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\theta).$$

Similarly, $\quad y_3 = r^3 e^{ax} \sin(bx + c + 3\theta).$

Hence, in general, $\quad y_n = \dfrac{d^n}{dx^n} [e^{ax} \sin(bx + c)] = r^n \cdot e^{ax} \sin(bx + c + n\theta)$

Putting values of r and θ,

$$\boxed{\mathbf{y_n = (a^2 + b^2)^{n/2} e^{ax} \cdot \sin\left(bx + c + n \cdot \tan^{-1} \frac{b}{a}\right)}}$$

IX. The n^{th} differential co-efficient of $e^{ax} \cos(bx + c)$

Let $\quad y = e^{ax} \cos(bx + c)$

then $\quad y_1 = a\, e^{ax} \cos(bx + c) - b\, e^{ax} \sin(bx + c)$

$$= e^{ax} [a \cos(bx + c) - b \sin(bx + c)]$$

As in Art. VIII, put $a = r \cos\theta$, and $b = r \sin\theta$.

$\therefore \qquad r = \sqrt{a^2 + b^2}$, $\tan\theta = \dfrac{b}{a}$, *i.e.*, $\theta = \tan^{-1}\dfrac{b}{a}$

$\therefore \qquad y_1 = e^{ax}\, r\, [\cos\theta \,.\, \cos(bx + c) - \sin\theta \sin(bx + c)]$

$= r\, e^{ax}\, [\cos(bx + c + \theta)]$.

Thus y_1 is obtained from y on multiplying it by the constant r and increasing the angle by the constant θ.

Repeating the same rule to y_1, we have

$$y_2 = r^2 e^{ax} \cos(bx + c + 2\theta).$$

Similarly, $\qquad y_3 = r^3 e^{ax} \cos(bx + c + 3\theta)$.

Hence, in general, $\qquad y_n = \dfrac{d^n}{dx^n}[e^{ax}\cos(bx + c)] = r^n e^{ax} \,.\, \cos(bx + c + n\theta)$

Putting values of r and θ,

$$\boxed{\mathbf{y_n = (a^2 + b^2)^{n/2}\, e^{ax} \,.\, \cos\left(bx + c + n \,.\, \tan^{-1}\frac{b}{a}\right)}}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the n^{th} differential co-efficient of log $(ax + x^2)$.*

Sol. Let $\qquad y = \log(ax + x^2) = \log x(a + x) = \log x + \log(x + a)$

Differentiating n times,

$$y_n = \frac{d^n}{dx^n}\log x + \frac{d^n}{dx^n}\log(x + a)$$

$$= \frac{(-1)^{n-1}(n-1)!\,.\,1^n}{x^n} + \frac{(-1)^{n-1}(n-1)!\,.\,1^n}{(x + a)^n} \qquad \text{[By Art. IV]}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x + a)^n}\right].$$

Example 2. *If $y = \cos^3 x$, find y_n.*

Sol. We know, $\qquad \cos 3x = 4\cos^3 x - 3\cos x$

$\therefore \qquad y = \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$

Differentiating n times,

$$y_n = \frac{1}{4}\left[\frac{d^n}{dx^n}\cos 3x + 3\frac{d^n}{dx^n}\cos x\right]$$

$$= \frac{1}{4}\left[3^n\cos\left(3x + \frac{n\pi}{2}\right) + 3.1^n\cos\left(x + \frac{n\pi}{2}\right)\right] \qquad \text{[By Art. VII]}$$

$$= \frac{1}{4}\left[3^n\cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right].$$

Example 3. *If* $y = \sin 2x \sin 3x$, *find* y_n.

Sol.
$$y = \sin 2x \sin 3x$$
$$= \tfrac{1}{2}[2 \sin 3x \sin 2x] = \tfrac{1}{2}[\cos x - \cos 5x]$$
$$[\because \ 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{d^n}{dx^n} \cos x - \frac{d^n}{dx^n} \cos 5x\right]$$
$$= \frac{1}{2}\left[\cos\left(x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right)\right].$$

Example 4. *Find* n^{th} *derivative of* $\sin^2 x \cos^3 x$.

Sol. Let
$$y = \sin^2 x \cos^3 x = (\sin x \cos x)^2 \cos x$$
$$= \tfrac{1}{4} (\sin 2x)^2 \cos x = \tfrac{1}{4} \sin^2 2x \cos x$$
$$= \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right) \cos x \qquad \left| \because \sin^2 \theta = \frac{1 - \cos 2\theta}{2}\right.$$
$$= \frac{1}{8} (\cos x - \cos 4x \cos x)$$

or
$$y = \frac{1}{16} [2 \cos x - (\cos 5x + \cos 3x)]$$
$$[\because \ 2 \cos A \cos B = \cos (A + B) + \cos (A - B)]$$

or
$$y = \frac{1}{16} [2 \cos x - \cos 5x - \cos 3x]$$

$$\therefore \quad y_n = \frac{1}{16}\left[2.1^n \cos\left(x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right) - 3^n \cos\left(3x + \frac{n\pi}{2}\right)\right]$$
$$= \frac{1}{16}\left[2 \cos\left(x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right) - 3^n \cos\left(3x + \frac{n\pi}{2}\right)\right].$$

Example 5. *If* $y = \dfrac{1}{1 - 5x + 6x^2}$, *find* y_n.

Sol.
$$y = \frac{1}{1 - 5x + 6x^2} = \frac{1}{(2x - 1)(3x - 1)} = \frac{2}{2x - 1} - \frac{3}{3x - 1}$$

$$\therefore \quad y_n = 2D^n \left(\frac{1}{2x - 1}\right) - 3D^n \left(\frac{1}{3x - 1}\right)$$
$$= 2\left[\frac{(-1)^n \, n!\,(2)^n}{(2x - 1)^{n+1}}\right] - 3\left[\frac{(-1)^n \, n!\,(3)^n}{(3x - 1)^{n+1}}\right] \qquad | \text{ By Art. (III)}$$
$$= (-1)^n \, n! \left[\frac{(2)^{n+1}}{(2x - 1)^{n+1}} - \frac{(3)^{n+1}}{(3x - 1)^{n+1}}\right].$$

5.3. USE OF PARTIAL FRACTIONS

In order to determine the n^{th} derivative, of any rational function, we have to split it into partial fractions. Forming partial fractions is converse process of taking L.C.M.

To resolve a fraction into partial fractions, the degree of numerator must be less than the degree of denominator.

ILLUSTRATIVE EXAMPLES

Example 1. *Find n^{th} derivative of* $\dfrac{1}{x^2 - a^2}$.

Sol. Let,
$$y = \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$\therefore$
$$y_n = \frac{1}{2a}\left[\frac{d^n}{dx^n}\left(\frac{1}{x-a}\right) - \frac{d^n}{dx^n}\left(\frac{1}{x+a}\right)\right]$$
$$= \frac{1}{2a}\left[\frac{(-1)^n n!\,1^n}{(x-a)^{n+1}} - \frac{(-1)^n n!.\,1^n}{(x+a)^{n+1}}\right]$$
$$= \frac{1}{2a}(-1)^n n!\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right].$$

Example 2. *If* $y = \dfrac{x^2}{(x-1)^2(x+2)}$, *find n^{th} derivative of y.*

Sol. To split y into partial fractions.

Let $x - 1 = z$, then
$$y = \frac{1}{z^2}.\frac{1+2z+z^2}{3+z}$$
$$= \frac{1}{z^2}\left(\frac{1}{3} + \frac{5z}{9} + \frac{4}{9}.\frac{z^2}{3+z}\right) \qquad | \text{ on division}$$
$$= \frac{1}{3z^2} + \frac{5}{9z} + \frac{4}{9(3+z)} = \frac{1}{3(x-1)^2} + \frac{5}{9(x-1)} + \frac{4}{9(x+2)}$$

Hence,
$$y_n = \frac{(-1)^n (n+1)!}{3(x-1)^{n+2}} + \frac{5(-1)^n n!}{9(x-1)^{n+1}} + \frac{4(-1)^n n!}{9(x+2)^{n+1}}.$$

Example 3. *Find the n^{th} derivative of* $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

Sol. We have, $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\tan^{-1} x$

Hence we have to find y_n if $y = 2\tan^{-1} x$.

Now,
$$y = 2\tan^{-1} x$$

$\therefore$
$$y_1 = \frac{2}{1+x^2} = \frac{2}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right] = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Hence,
$$y_n = \frac{1}{i}(-1)^{n-1}.(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$
| Differentiating $(n-1)$ times

Suppose $x = r\cos\theta$, and $1 = r\sin\theta$ so that

$$r^2 = x^2 + 1 \text{ and } \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Thus, $$y_n = \frac{(-1)^{n-1}(n-1)!}{ir^n}\{(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\}$$

$\Rightarrow$ $$y_n = \frac{(-1)^{n-1}(n-1)!}{i\,.\,r^n}\,.\,2i\sin n\theta \qquad | \text{ Using De Moivre's theorem}$$

$\Rightarrow$ $$y_n = 2\,(-1)^{n-1}\,(n-1)!.\,\sin n\theta\,.\,\sin^n\theta. \qquad \text{where} \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Example 4. *Prove that the value of the n^{th} differential co-efficient of $\frac{x^3}{x^2-1}$ for $x = 0$ is zero, if n is even ; and is $-(n\,!)$ if n is odd and greater than 1.*

Sol. Let $$y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$= x + \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right] \qquad \ldots(1)$$

If $n > 1$, the second and higher derivatives of x are 0.

$\therefore$ $$y_n = \frac{1}{2}\left[\frac{(-1)^n\,n!}{(x-1)^{n+1}} + \frac{(-1)^n\,n!}{(x+1)^{n+1}}\right]$$

$$= \frac{1}{2}\,(-1)^n\,n!\left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}}\right] \qquad \ldots(2)$$

Case I. When n is even, putting $x = 0$ in (2),

$$(y_n)_0 = \frac{(1)\,n!}{2}\left[\frac{1}{1} + \frac{1}{(-1)}\right] = \frac{n!}{2}\,[1-1] = 0.$$

Case II. When n is odd

$$(y_n)_0 = \frac{(-1)\,n!}{2}\left[\frac{1}{1} + \frac{1}{(1)}\right] = -\frac{(n!)}{2}\,[1+1] = -\,n\,!.$$

Example 5. *If $y = x\log\frac{x-1}{x+1}$, show that* $y_n = (-1)^{n-2}\,(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

Sol. $$y = x\,[\log(x-1) - \log(x+1)] \qquad \ldots(1)$$

Differentiating y w.r.t. x, we get

$$y_1 = x\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \log(x-1) - \log(x+1)$$

$$= \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1) \qquad \ldots(2)$$

Again, Differentiating (2), $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{(-1)^{n-1}\,(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}\,(n-1)!}{(x+1)^n} + \frac{(-1)^{n-2}\,(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\,(n-2)!}{(x+1)^{n-1}}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{(-1)(n-1)}{(x-1)^n}+\frac{(-1)(n-1)}{(x+1)^n}+\frac{x-1}{(x-1)^n}-\frac{x+1}{(x+1)^n}\right]$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right].$$

5.4. LEIBNITZ THEOREM

This theorem helps us to find the n^{th} differential co-efficient of the product of two functions in terms of the successive derivatives of the functions.

Statement. *If u and v are two functions of x, having derivatives of the n^{th} order, then*

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots\ldots + {}^nC_r\, u_{n-r} v_r + \ldots + {}^nC_n\, uv_n,$$

where suffixes of u and v denote differentiations w.r.t. x.

Proof. We shall prove the theorem by Mathematical Induction.

Step I. Let $y = uv$

By actual differentiation, we have

$$y_1 = u_1v + uv_1$$

and

$$y_2 = u_2v + u_1v_1 + u_1v_1 + uv_2$$

$$= u_2v + 2u_1v_1 + uv_2 = u_2v + {}^2C_1\, u_1v_1 + {}^2C_2\, uv_2.$$

Thus, the theorem is true for $n = 1, 2$.

Step II. Let us assume that the theorem is true for a particular value of n, say m, so that, we have

$$y_m = u_m v + {}^mC_1\, u_{m-1} v_1 + {}^mC_2\, u_{m-2} v_2 + \ldots\ldots$$
$$+ {}^mC_{r-1}\, u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \ldots\ldots + {}^mC_m\, uv_m.$$

Step III. Differentiating both sides, we have

$$y_{m+1} = u_{m+1} v + u_m v_1 + {}^mC_1\, u_m v_1 + {}^mC_1\, u_{m-1} v_2 + {}^mC_2\, u_{m-1} v_2$$
$$+ {}^mC_2\, u_{m-2} v_3 + \ldots\ldots + {}^mC_{r-1}\, u_{m-r+2} v_{r-1} + {}^mC_{r-1}\, u_{m-r+1} v_r$$
$$+ {}^mC_r\, u_{m-r+1} v_r + {}^mC_r\, u_{m-r} v_{r+1} + \ldots\ldots$$
$$+ {}^mC_m\, u_1v_m + {}^mC_m\, uv_{m+1}.$$
$$= u_{m+1} v + ({}^mC_1 + 1)v_m v_1 + ({}^mC_2 + {}^mC_1)\, u_{m-1} v_2 + \ldots\ldots$$
$$+ ({}^mC_r + {}^mC_{r-1})\, u_{m-r+1} v_r + \ldots\ldots + {}^mC_m\, uv_{m+1}.$$

But, we know that ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

Putting $r = 1, 2, 3, \ldots\ldots$

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1 \text{ or } 1 + {}^mC_1 = {}^{m+1}C_1$$
$${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

....

and

$${}^mC_m = 1 = {}^{m+1}C_{m+1}.$$

$$\therefore \quad y_{m+1} = u_{m+1} v + {}^{m+1}C_1\, u_m v_1 + {}^{m+1}C_2\, u_{m-1} v_2 + \ldots\ldots$$
$$+ {}^{m+1}C_r\, u_{m-r+1} v_r + \ldots\ldots + {}^{m+1}C_{m+1}\, uv_{m+1}$$

which is of exactly the same form as the given formula with $n = m + 1$.

$\therefore$ it is clear that if the theorem is true for $n = m$, then it is also true for the next higher value $n = m + 1$.

But in Step I, we have proved that the theorem is true for $n = 2$.

$\therefore$ it must be true for next higher values $n = 2 + 1 = 3$.

Again $\because$ the theorem is true for $n = 3$.

$\therefore$ it must be true for next higher value $n = 3 + 1$, *i.e.,* 4 and so on.

Hence the theorem is true for any positive integer n.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the n^{th} derivative of $x^2 \sin x$ at $x = 0$.*

Sol. Let $u = \sin x$ and $v = x^2$

$$\therefore \quad u_n = \sin\left(x + n\frac{\pi}{2}\right) \qquad v_1 = 2x$$

$$u_{n-1} = \sin\left[x + (n-1)\frac{\pi}{2}\right] \qquad v_2 = 2$$

$$u_{n-2} = \sin\left[x + (n-2)\frac{\pi}{2}\right] \qquad v_3 = 0$$

Now by Leibnitz Theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2$$

or
$$\frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + n\frac{\pi}{2}\right) . x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right] 2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right] 2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1) \sin\left[x + (n-2)\frac{\pi}{2}\right].$$

At $x = 0$, $\quad y_n = n(n-1) \sin (n-2)\dfrac{\pi}{2} = (n - n^2) \sin \dfrac{n\pi}{2}$

Note. Generally, we take x^n as v.

Example 2. *Differentiate n times the differential equations:*

$$(i)\ (1 - x^2) \frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0 \qquad (ii)\ x^2y_2 + xy_1 + y = 0.$$

Sol. (i) Differentiating each term of the given equation n times by Leibnitz theorem, we get

$$\{y_{n+2}\,(1 - x^2) + {}^nC_1\, y_{n+1}\,(-2x) + {}^nC_2\, y_n\,(-2)\} - \{y_{n+1}\,(x) + {}^nC_1\, y_n\,(1)\} + a^2\, y_n = 0$$

$$\Rightarrow \quad (1 - x^2)\, y_{n+2} + (-2nx - x)y_{n+1} + \{-n(n-1) - n + a^2\}\, y_n = 0$$

$$\Rightarrow \quad (1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 - a^2)\, y_n = 0$$

(ii) Differentiating each term of the given equation n times by Leibnitz theorem, we get

$$\{y_{n+2}\,(x^2) + {}^nC_1\, y_{n+1}\,(2x) + {}^nC_2\, y_n\,(2)\} + \{y_{n+1}(x) + {}^nC_1\, y_n(1)\} + y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2nx + x)\, y_{n+1} + \{n(n-1) + n + 1\}\, y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 + 1)\, y_n = 0.$$

Example 3. *If $y = \sin (m \sin^{-1} x)$, then prove that*

$$(1-x^2)y_2 - xy_1 + m^2y = 0$$

and
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0.$$

Sol.
$$y = \sin (m \sin^{-1} x) \qquad ...(1)$$

$\therefore$
$$y_1 = \cos (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

Cross-multiplying $\sqrt{1-x^2}\, y_1 = m \cos (m \sin^{-1} x)$.

Squaring both sides,

$$(1-x^2)y_1^2 = m^2 \cos^2 (m \sin^{-1} x)$$
$$= m^2[1 - \sin^2 (m \sin^{-1} x)] \qquad \left| \because \cos^2\theta = 1 - \sin^2\theta \right.$$
$$= m^2(1-y^2) \qquad \text{[By (1)]}$$

Again differentiating both sides w.r.t. x,

$$(1-x^2)\, 2y_1y_2 - 2xy_1^2 = -2m^2yy_1$$

Dividing every term by $2y_1$,

$$(1-x^2)y_2 - xy_1 = -m^2y$$

or
$$(1-x^2)y_2 - xy_1 + m^2y = 0.$$

Now differentiating every term n times by Leibnitz Theorem, we have

$$[y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)] - [y_{n+1} \cdot x + {}^nC_1\, y_n] + m^2y_n = 0$$

or
$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$\Rightarrow$
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Example 4. *If $y^{1/m} + y^{-1/m} = 2x$, prove that*

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Sol. $\quad y^{1/m} + y^{-1/m} = 2x$

Put $\quad y^{1/m} = z$

$\therefore$
$$z + \frac{1}{z} = 2x \quad \text{or} \quad z^2 - 2xz + 1 = 0$$

$\therefore$
$$z = \frac{2x \pm 2\sqrt{x^2-1}}{2} = x \pm \sqrt{x^2-1}$$

i.e.,
$$y^{1/m} = x \pm \sqrt{x^2-1} \quad \text{or} \quad y = (x \pm \sqrt{x^2-1})^m$$

If
$$y = (x + \sqrt{x^2-1})^m, \text{ then}$$

$$y_1 = m(x + \sqrt{x^2-1})^{m-1}\left(1 + \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-1}} \cdot 2x\right)$$
$$= m(x + \sqrt{x^2-1})^{m-1}\left(\frac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}\right) = \frac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

And, similarly, if $\quad y = (x - \sqrt{x^2-1})^m$,

$$y_1 = \frac{-my}{\sqrt{x^2-1}}.$$

In either case, squaring, we get

$$y_1^2 = \frac{m^2y^2}{x^2-1} \quad \text{or} \quad (x^2-1)\,y_1^2 = m^2y^2.$$

Differentiating again,

$$(x^2-1)\,.\,2y_1y_2 + y_1^2\,.\,2x = m^2\,.\,2yy_1$$

Dividing both sides by $2y_1$, we have

$$(x^2-1)y_2 + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz Theorem, we get

$$[y_{n+2}\,(x^2-1) + n\,.\,y_{n+1}\,.\,2x + \frac{n(n-1)}{2!}\,.\,y_n\,.\,2] + [y_{n+1}\,.\,x + ny_n\,.\,1] - m^2y_n = 0$$

or $$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-n+n-m^2)y_n = 0$$

or $$(x^2-1)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)y_n = 0.$$

Example 5. *If $y = a\cos(\log x) + b\sin(\log x)$, prove that*

(i) $x^2y_2 + xy_1 + y = 0$ and *(ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.*

Sol. $$y = a\cos(\log x) + b\sin(\log x) \quad \ldots(1)$$

$\therefore$ $$y_1 = -a\sin(\log x)\,.\,\frac{1}{x} + b\cos(\log x)\,.\,\frac{1}{x}$$

$\Rightarrow$ $$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Again, $$xy_2 + y_1 = -a\cos(\log x)\,.\,\frac{1}{x} - b\sin(\log x)\,.\,\frac{1}{x}$$

$\Rightarrow$ $$x^2y_2 + xy_1 = -y$$

$\Rightarrow$ $$x^2y_2 + xy_1 + y = 0 \quad \ldots(2)$$

Differentiating equation (2) n times by Leibnitz Theorem, we get

$$\{y_{n+2}\,.\,(x^2) + {}^nC_1\,y_{n+1}\,.\,(2x) + {}^nC_2\,y_n\,.\,(2)\} + \{y_{n+1}\,.\,(x) + {}^nC_1\,.\,y_n\,(1)\} + y_n = 0$$

$\Rightarrow$ $$x^2\,y_{n+2} + y_{n+1}\,(2nx + x) + y_n\,\{n(n-1) + n + 1\} = 0$$

$\Rightarrow$ $$x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+1)\,y_n = 0$$

Hence the results.

Example 6. *If $y = (x^2-1)^n$, prove that $(x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)\,y_n = 0$.*

Sol. $$y = (x^2-1)^n \quad \ldots(1)$$

$\therefore$ $$y_1 = n\,(x^2-1)^{n-1}\,.\,2x$$

$\Rightarrow$ $$(x^2-1)\,y_1 = 2nxy \quad \ldots(2) \quad |\ \text{Using (1)}$$

Differentiating again, we get

$$(x^2-1)\,y_2 + 2xy_1 = 2n\,(xy_1 + y)$$

$\Rightarrow$ $$(x^2-1)\,y_2 + 2(1-n)\,xy_1 - 2ny = 0$$

Differentiating n times by Leibnitz theorem, we get

$$\{y_{n+2}\,(x^2-1) + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n\,(2)\} + 2(1-n)\,\{y_{n+1}\,(x) + {}^nC_1\,y_n(1)\} - 2ny_n = 0$$

$\Rightarrow$ $$(x^2-1)\,y_{n+2} + (2nx + 2x - 2nx)\,y_{n+1} + (n^2 - n + 2n - 2n^2 - 2n)\,y_n = 0$$

$\Rightarrow$ $$(x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

Example 7. *(i) If $x = \tan(\log y)$, prove that*

$$(1 + x^2)\, y_{n+1} + (2nx - 1)\, y_n + n(n-1)\, y_{n-1} = 0$$

(ii) If $y = \sin \log_e (x^2 + 2x + 1)$, prove that

$$(1 + x)^2\, y_{n+2} + (2n+1)(1+x)\, y_{n+1} + (n^2 + 4)\, y_n = 0.$$

Sol. (i) $\qquad x = \tan(\log y)$

$\Rightarrow \qquad y = e^{\tan^{-1} x}$ ...(1)

$\therefore \qquad y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$

$\Rightarrow \qquad (1 + x^2)\, y_1 = y$ ...(2) | Using (1)

Differentiating (2) n times by Leibnitz theorem, we get

$$y_{n+1}(1 + x^2) + {}^nC_1\, y_n (2x) + {}^nC_2\, y_{n-1}(2) = y_n$$

$\Rightarrow \qquad (1 + x^2)\, y_{n+1} + (2nx - 1)\, y_n + n(n-1)\, y_{n-1} = 0.$

(ii) $\qquad y = \sin \log_e (x^2 + 2x + 1)$ ...(1)

$\therefore \qquad y_1 = \cos \log_e (x^2 + 2x + 1) \,.\, \dfrac{1}{x^2 + 2x + 1}\,(2x + 2)$

$\qquad = \cos \log_e (x^2 + 2x + 1) \,.\, \dfrac{2}{x+1}$

$\Rightarrow \qquad (x + 1)\, y_1 = 2 \cos \log_e (x^2 + 2x + 1)$

Squaring both sides,

$$(x + 1)^2\, y_1^{\,2} = 4 \cos^2 \log_e (x^2 + 2x + 1) = 4(1 - y^2) \qquad ...(2)$$

Differentiating (2) again w.r.t. x, we get

$$(x + 1)^2 \,.\, 2y_1y_2 + y_1^{\,2} \,.\, 2(x + 1) = 4(-2yy_1)$$

$\Rightarrow \qquad (x + 1)^2\, y_2 + (x + 1)y_1 + 4y = 0$ ...(3)

Differentiating (3) n times by Leibnitz theorem, we get

$$\{y_{n+2}(x + 1)^2 + {}^nC_1\, y_{n+1} \,.\, 2(x + 1) + {}^nC_2\, y_n \,.\, 2\} + \{y_{n+1}(x + 1) + {}^nC_1\, y_n (1)\} + 4y_n = 0$$

$\Rightarrow \qquad (x + 1)^2\, y_{n+2} + (2nx + 2n + x + 1)\, y_{n+1} + (n^2 - n + n + 4)\, y_n = 0$

$\Rightarrow \qquad (1 + x)^2\, y_{n+2} + (2n + 1)(1 + x)\, y_{n+1} + (n^2 + 4)\, y_n = 0.$

Example 8. *If $y = x^n \log x$ prove that*

(a) $y_{n+1} = \dfrac{n!}{x}$ $\qquad$ (b) $x^2y_{p+2} + (2p - 2n + 1)\, xy_{p+1} + (p - n)^2\, y_p = 0.$

Sol. (a) $\qquad y = x^n \log x$ ...(1)

$\therefore \qquad y_1 = x^n \left(\dfrac{1}{x}\right) + nx^{n-1} \log x$

$\Rightarrow \qquad xy_1 = x^n + nx^n \log x = x^n + ny$ ...(2) | Using (1)

Differentiating (2) n times, we get

$y_{n+1} \,.\, (x) + {}^nC_1\, y_n (1) = n! + ny_n$

$\Rightarrow \qquad y_{n+1}(x) = n!$

or $\qquad y_{n+1} = \dfrac{n!}{x}$

(b) $$y_1 = x^n \cdot \frac{1}{x} + nx^{n-1} \log x \qquad | \text{ Diff. } y \text{ w.r.t. } x$$

$$\Rightarrow \quad xy_1 = x^n + nx^n \log x \qquad | \text{ Multiplying by } x$$

$$\Rightarrow \quad xy_1 = x^n + ny \qquad ...(1)$$

Differentiating equation (1) w.r.t. x, we get

$$xy_2 + y_1 = nx^{n-1} + ny_1$$

$$xy_2 + (1-n)\, y_1 = nx^{n-1}$$

$$x^2y_2 + (1-n)\, xy_1 = nx^n = n\,(xy_1 - ny) \qquad | \text{ Using (1)}$$

$$\Rightarrow \quad x^2y_2 + (1-2n)\, xy_1 + n^2y = 0 \qquad ...(2)$$

Differentiating equation (2), p times by Leibnitz theorem,

$$[y_{p+2}\,(x^2) + {}^pC_1\, y_{p+1}\,(2x) + {}^pC_2 y_p(2)] + (1-2n)\,[y_{p+1}\,(x) + {}^pC_1\, y_p(1)] + n^2y_p = 0$$

$$\Rightarrow \quad x^2 y_{p+2} + y_{p+1}\,\{2px + (1-2n)\,x\} + y_p\{p\,(p-1) + (1-2n)\,p + n^2\} = 0$$

$$\Rightarrow \quad x^2 y_{p+2} + (2p - 2n + 1)\, xy_{p+1} + (p-n)^2\, y_p = 0.$$

TEST YOUR KNOWLEDGE

1. If $y = \sin^{-1} x$, prove that $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} - n^2y_n = 0$.

2. (a) If $y = e^{m \sin^{-1} x}$, prove that:

(i) $(1-x^2)\, y_2 - xy_1 = m^2y$ (ii) $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+m^2)y_n = 0$.

(b) If $x = \sin\left(\frac{\log y}{a}\right)$, prove that $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+a^2)\, y_n = 0$.

3. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\, y_{n+2} - (2n+3)\, x\, y_{n+1} - (n+1)^2\, y_n = 0$.

4. If $y = e^{\tan^{-1} x}$, prove that $(1+x^2)\, y_{n+2} + [(2n+2)x - 1]y_{n+1} + n(n+1)y_n = 0$.

5. If $y = \cos(\log x)$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

6. If $y = x \cos(\log x)$, prove that $x^2\, y_{n+2} + (2n-1)\, xy_{n+1} + (n^2 - 2n + 2)y_n = 0$.

7. If $y = (\sin^{-1} x)^2$, prove that

(i) $(1-x^2)\, y_2 - xy_1 - 2 = 0$ (ii) $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} - n^2y_n = 0$.

8. If $y = (\tan^{-1} x)^2$, prove that $(x^2+1)^2\, y_2 + 2x(1+x^2)\, y_1 = 2$

9. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2\, y_{n+2} + (2n+1)\, xy_{n+1} + 2n^2y_n = 0$

10. If $y = \cos(m \sin^{-1} x)$, prove that $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} - (n^2-m^2)y_n = 0$

11. If $y = \sin(m \sinh^{-1} x)$, prove that $(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+m^2)y_n = 0$

12. If $y = (\sinh^{-1} x)^2$, prove that $(1+x^2)\, y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$.

13. If $y = \sqrt{1-x^2}\, \sin^{-1} x$, prove that

(i) $y_3\,(1-x^2) - 3y_2\, x + 2 = 0$ (ii) $y_{n+3}\,(1-x^2) - (2n+3)\, xy_{n+2} - n(n+2)\, y_{n+1} = 0$.

14. If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n\, n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right]$.

15. If $x + y = 1$, prove that $\frac{d^n}{dx^n}(x^n y^n) = n!\,[y^n - ({}^nC_1)^2 y^{n-1} . x + ({}^nC_2)^2 y^{n-2} . x^2 - ... + (-1)^n x^n]$.

16. If $x \sin\phi + y\cos\phi = a$ and $x\cos\phi - y\sin\phi = b$, show that $\frac{d^m x}{d\phi^m} . \frac{d^n y}{d\phi^n} - \frac{d^m y}{d\phi^m} . \frac{d^n x}{d\phi^n}$ = constant.

17. If $y = A(x + \sqrt{x^2 + a^2})^n + B(x + \sqrt{x^2 + a^2})^{-n}$, prove that
$(x^2 + a^2)y_{m+2} + (2m+1)xy_{m+1} + (m^2 - n^2)y_m = 0$

18. Evaluate $D^n[\log(x^2 - 2x\cos\alpha + 1)]$ hence or otherwise find $D^n\left[\tan^{-1}\left(\frac{x\sin\alpha}{1 - x\cos\alpha}\right)\right]$.

19. If $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$, prove that
(*a*) $(x^2 - 1)y_2 + xy_1 - n^2 y = 0$ (*b*) $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} = 0$.

20. If $y\sqrt{x^2 - 1} = \log(x + \sqrt{x^2 - 1})$, prove that $(x^2 - 1)y_{n+1} + (2n+1)xy_n + n^2 y_{n-1} = 0$.

Answer

18. $\frac{(-1)^{n-1}(n-1)!}{(x^2 - 2x\cos\alpha + 1)^{n/2}}\left[2\cos\left(n\tan^{-1}\frac{\sin\alpha}{x - \cos\alpha}\right)\right]$; $\frac{(-1)^{n-1}(n-1)!}{(x^2 - 2x\cos\alpha + 1)^{n/2}}\sin\left(n\tan^{-1}\frac{\sin\alpha}{x - \cos\alpha}\right)$.

5.5. DETERMINATION OF THE VALUE OF THE n^{TH} DERIVATIVE OF A FUNCTION AT $x = 0$

Sometimes it is required to find the n^{th} derivative of a function for $x = 0$.

The **working rule** to find $(y_n)_{x=0}$ is being given below:

1. Put the given function equal to y.

2. Find $y_1 = \frac{dy}{dx}$.

Then (*i*) Take L.C.M. (if possible).

(*ii*) Square both sides if square roots are there.

(*iii*) Try to get y in R.H.S. (if possible).

3. Again differentiate both sides w.r.t. x to get an equation in y_2, y_1, y.

4. Differentiate both sides n times w.r.t. x by Leibnitz Theorem.

5. Put $x = 0$ in equations of steps 1, 2, 3, 4.

6. Put $n = 1, 2, 3, 4$ in last equation of step 5.

7. Discuss the two cases when n is even and when n is odd.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $y = e^{m\cos^{-1}x}$, *show that*

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0 \text{ and calculate } y_n(0).$$

Sol. Here
$$y = e^{m\cos^{-1}x} \quad ...(1)$$

$\therefore$
$$y_1 = e^{m\cos^{-1}x} . \frac{-m}{\sqrt{1 - x^2}} = -\frac{my}{\sqrt{1 - x^2}} \quad ...(2)$$

Squaring both sides, we have

$$(1-x^2)\,y_1^{\,2} = m^2y^2$$

Differentiating w.r.t. x, we have

$$(1-x^2)\,2y_1y_2 - 2xy_1^{\,2} = 2m^2yy_1$$

Dividing by $2y_1$, we get

$$(1-x^2)\,y_2 - xy_1 = m^2y \qquad ...(3)$$

Differentiating n times by Leibnitz Theorem, we get

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = m^2y_n$$

i.e.,

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \qquad ...(4)$$

By putting $x = 0$, in (1), (2), (3) and (4), we get

$$y(0) = e^{m\,.\,\pi/2}$$

$$y_1(0) = -m\,.\,e^{m\,.\,\pi/2}$$

$$y_2(0) = m^2\,.\,y(0) = m^2\,.\,e^{m\,.\,\pi/2}$$

$$y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad ...(5)$$

Putting $n = 1, 2, 3, 4, ...$ in (5), we have

$$y_3(0) = (1^2+m^2)y_1(0) = -m(1^2+m^2)e^{m\,.\,\pi/2}$$

$$y_4(0) = (2^2+m^2)y_2(0) = m^2(2^2+m^2)e^{m\,.\,\pi/2}$$

$$y_5(0) = (3^2+m^2)y_3(0) = -m(1^2+m^2)(3^2+m^2)e^{m\,.\,\pi/2}$$

$$y_6(0) = (4^2+m^2)y_4(0) = m^2(2^2+m^2)(4^2+m^2)e^{m\,.\,\pi/2}$$

In general, $y_n(0) = \begin{cases} -m\,.\,e^{m\,.\,\pi/2}\,(1^2+m^2)(3^2+m^2)\,...\,[(n-2)^2+m^2] & \text{when } n \text{ is odd.} \\ m^2\,.\,e^{m\,.\,\pi/2}\,(2^2+m^2)(4^2+m^2)\,...\,[(n-2)^2+m^2], & \text{when } n \text{ is even.} \end{cases}$

Example 2. *If $y = (\sin^{-1} x)^2$, prove that*

(i) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$ *(ii)* $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2\,y_n = 0.$

Also find the value of n^{th} derivative of y for $x = 0$.

Sol. $y = (\sin^{-1} x)^2$

Differentiating both sides w.r.t. x,

$$y_1 = 2(\sin^{-1} x)\,.\,\frac{1}{\sqrt{1-x^2}} \qquad ...(1)$$

Squaring both sides, $(1-x^2)y_1^{\,2} = 4\,(\sin^{-1} x)^2 = 4y.$

Differentiating again, $2(1-x^2)\,y_1y_2 - 2xy_1^{\,2} = 4y_1$

Dividing both sides by $2y_1$, $(1-x^2)\,y_2 - xy_1 - 2 = 0 \qquad ...(2)$

Differentiating n times by Leibnitz Theorem,

$$(1-x^2)y_{n+2} + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2) - xy_{n+1} - {}^nC_1\,y_n = 0$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \qquad ...(3)$$

Put $x = 0$ in (1), (2) and (3), then

$$y_1(0) = 0 \quad \text{and} \quad y_2(0) = 2,$$

and

$$y_{(n+2)}(0) = n^2y_n(0) \qquad ...(4)$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$ in (4), we get

$$y_3(0) = 1^2 \,.\, y_1(0) = 0$$
$$y_4(0) = 2^2 \,.\, y_2(0) = 2 \,.\, 2^2$$
$$y_5(0) = 3^2 \,.\, y_3(0) = 0$$
$$y_6(0) = 4^2 \,.\, y_4(0) = 2 \,.\, 2^2 \,.\, 4^2$$
$$y_7(0) = 5^2 \,.\, y_5(0) = 0$$
$$y_8(0) = 6^2 \,.\, y_6(0) = 2 \,.\, 2^2 \,.\, 4^2 \,.\, 6^2.$$

....

....

In general,

When n is odd, $y_n(0) = 0$.

When n is even, $y_n(0) = 2 \,.\, 2^2 \,.\, 4^2 \,.\, 6^2 \ldots\ldots (n-2)^2$.

Example 3. *If $y = \tan^{-1} x$, prove that $(1 + x^2)y_{n+1} + 2nxy_n + n(n-1)\, y_{n-1} = 0$. Hence determine the values of all the derivatives of y with respect to x when $x = 0$.*

Sol. Here $$y = \tan^{-1} x \qquad \ldots(1)$$

$\therefore$ $$y_1 = \frac{1}{1+x^2} \qquad \ldots(2)$$

or $$y_1(1 + x^2) = 1$$

Differentiating n times by Leibnitz Theorem, we have

$$y_{n+1}(1 + x^2) + ny_n \,.\, 2x + \frac{n(n-1)}{2} y_{n-1} \,.\, 2 = 0$$

or $$(1 + x^2)\, y_{n+1} + 2nxy_n + n(n-1)\, y_{n-1} = 0 \qquad \ldots(3)$$

Putting $x = 0$ in (1), (2) and (3), we have

$$y(0) = 0,\ y_1(0) = 1$$

and $$y_{n+1}(0) = -\,n(n-1)y_{n-1}(0) \qquad \ldots(4)$$

Putting $n = 1, 2, 3, 4, \ldots\ldots\ldots$ in (4), we get

$$y_2(0) = -\,1 \,.\, (0) \,.\, y(0) = 0$$
$$y_3(0) = -\,2 \,.\, (1) \,.\, y_1(0) = -\,2 = (-\,1)^1\, 2\,!$$
$$y_4(0) = -\,3 \,.\, (2) \,.\, y_2(0) = 0$$
$$y_5(0) = -\,4 \,.\, (3) \,.\, y_3(0) = -\,4 \,.\, (3) \,.\, (-\,2) = (-\,1)^2\, 4\,!$$

....

In general,

When n is even, $y_n(0) = 0$.

When n is odd, $y_n(0) = (-\,1)^{\frac{n-1}{2}}\, (n-1)\,!$.

TEST YOUR KNOWLEDGE

1. If $y = \sin^{-1} x$, prove that $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$
   Also find the value of y_n when $x = 0$.
2. If $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$, prove that $y_n = (n - 1)^2\, y_{n-2}$ for $x = 0$.
3. If $y = [x + \sqrt{1 + x^2}\,]^m$, find $y_n\,(0)$.
4. If $y = \sin\,(a \sin^{-1} x)$, find $y_n(0)$.
5. Prove that the value when $x = 0$ of $D^n\,(\tan^{-1} x)$ is 0, $(n - 1)\,!$ or $-\,(n - 1)\,!$ according as n is of the form $2p$, $4p + 1$ or $4p + 3$ respectively.
6. If $\log y = \tan^{-1} x$, show that $(1 + x^2)\, y_{n+2} + \{2(n + 1)\, x - 1\}\, y_{n+1} + n\,(n + 1)\, y_n = 0$ and hence find y_3, y_4 and y_5 at $x = 0$.

Answers

1. When n is even, $y_n(0) = 0$
   When n is odd, $y_n(0) = 1^2 \,.\, 3^2 \,.\, 5^2 \,\ldots\ldots\, (n - 2)^2$
3. When n is even, $y_n(0) = m^2\,(m^2 - 2^2)(m^2 - 4^2) \,\ldots\ldots\, [m^2 - (n - 2)^2]$
   When n is odd, $y_n(0) = m(m^2 - 1^2)(m^2 - 3^2) \,\ldots\ldots\, [m^2 - (n - 2)^2]$
4. When n is even, $y_n(0) = 0$
   When n is odd, $y_n(0) = a(1^2 - a^2)(3^2 - a^2) \,\ldots\ldots\, [(n - 2)^2 - a^2]$.
6. $(y_3)_0 = -\,1$, $(y_4)_0 = -\,7$, $(y_5)_0 = 5$.

PARTIAL DIFFERENTIATION

5.6. FUNCTION OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y and this is denoted by $z = f(x, y)$.

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined when its base and altitude are known. Thus, area of a triangle is a function of two variables, base and altitude.

(In a similar way, a function of more than two variables can be defined).

Geometrically, Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane. Then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space, we get a surface.

Hence the function z = f(x, y) represents a surface.

5.7. CONTINUITY

A function $f\,(x, y)$ is said to be continuous at a point (a, b) if, for any arbitrarily chosen positive number ε, however small, we can find a corresponding number δ such that $|\; f(x, y)$

$- f(a, b) \mid < \varepsilon$ for every point (x, y) within the circle with its centre at (a, b) and radius δ. *i.e.*, whenever $0 < (x-a)^2 + (y-b)^2 < \delta^2$.

Alternatively, $f(x, y)$ is said to be continuous at (a, b) if $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = f(a, b)$ irrespective of the path along which $x \to a, y \to b$.

It should not be assumed that the path along which the point (x, y) tends to (a, b) is immaterial, because $\underset{x \to a}{\text{Lt}} \{ \underset{y \to b}{\text{Lt}} f(x, y)\}$ is not always equal to $\underset{y \to b}{\text{Lt}} \{ \underset{x \to a}{\text{Lt}} f(x, y)\}$.

However, *usually*, the limit is the same irrespective of the path of approach.

In what follows, we shall assume that the functions considered are continuous and their partial differential co-efficients as defined in the next section, exist.

5.8. PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z, with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or f_x.

Thus, $$\frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z, with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or f_y

Thus, $$\frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. that variable, **treating all other variables as constant.**]

Geometrically, Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 5.6, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore \quad z = f(x, y)$ and $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\dfrac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\dfrac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\dfrac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

5.9. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad f_{xx}\,; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad f_{xy}\,; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{yx}$$

In general, $$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{xy}=f_{yx}.$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$.

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots\ldots, x_n$, we get

$$\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots, \frac{\partial z}{\partial x_n}.$$

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\,; \quad \frac{\partial z}{\partial y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

(*ii*) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(uv)=u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(uv)=u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}$$

(*iii*) If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}}{v^2}, \quad \frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(\frac{u}{v}\right)=\frac{v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}}{v^2}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the first order partial derivatives of the following:*

(*i*) $u = tan^{-1}\frac{x^2+y^2}{x+y}$ (*ii*) $u = cos^{-1}\left(\frac{x}{y}\right)$.

Sol. (*i*) $$u = \tan^{-1}\frac{x^2+y^2}{x+y}$$

Differentiating u partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2+(x^2+y^2)^2} \cdot \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2)-(x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2+(x^2+y^2)^2}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2} \qquad \text{...(1)}$$

Since u is symmetrical w.r.t. x and y, $\frac{\partial u}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$

(*ii*) $$u = \cos^{-1}\left(\frac{x}{y}\right)$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} = \frac{-1}{\sqrt{y^2-x^2}}$$

and $$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}}\left(-\frac{x}{y^2}\right) = \frac{x}{y\sqrt{y^2-x^2}}.$$

Example 2. *If* $u = x^y$, *show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol. $$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = yx^{y-1}\log x + x^y . \frac{1}{x} = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y \log x + 1)] \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1} + yx^{y-1}\log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y \log x + 1)] \qquad \text{...(2)}$$

From (1) and (2), $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 3. *If* $z = e^{ax+by} f(ax - by)$, *show that* $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz.$

Sol. We have, $\quad z = e^{ax+by} . f(ax - by) \quad$ ...(1)

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial z}{\partial x} = e^{ax+by} . f'(ax - by) . a + f(ax - by) . e^{ax+by} . a$$

$$\therefore \quad b\frac{\partial z}{\partial x} = ab\, e^{ax+by}\, [f'(ax - by) + f(ax - by)] \quad ...(2)$$

Again differentiating (1) w.r.t. y, partially, we get

$$\frac{\partial z}{\partial y} = e^{ax+by} . f'(ax - by) . (-b) + f(ax - by) . e^{ax+by} . b$$

$$a\frac{\partial z}{\partial y} = ab\, e^{ax+by}\, [-f'(ax - by) + f(ax - by)] \quad ...(3)$$

Adding (2) and (3), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Example 4. *If* $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *show that*

(*i*) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = -\dfrac{9}{(x+y+z)^2}.$

(*ii*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} + 2\dfrac{\partial^2 u}{\partial y \partial z} + 2\dfrac{\partial^2 u}{\partial z \partial x} + 2\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{-9}{(x+y+z)^2}.$

Sol. (*i*) $\quad u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz};\ \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz};\ \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

Adding, $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \dfrac{3}{x+y+z}$

$[\because \ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)]$

Now $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right) u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2} \quad ...(1)$$

(*ii*) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right) u = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y}$$

$$\left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

[from (1)]

Example 5. *If* $\theta = t^n e^{-\frac{r^2}{4t}}$, *find the value of n which will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

Sol.

$$\theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial r} = t^n \cdot e^{-\frac{r^2}{4t}} \cdot \left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 \cdot t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right)$$

Also,

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} \cdot \left(\frac{r^2}{4t^2}\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

Since, $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$ [Given]

$$\therefore \quad \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \quad \frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t}$$

$$\therefore \quad n = -\frac{3}{2}.$$

Example 6. *If* $x^x y^y z^z = c$, *show that at* $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking log, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z}\frac{\partial z}{\partial y} + 1 \cdot \log z \cdot \frac{\partial z}{\partial y} = 0$$

or

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0 \qquad ...(1)$$

or
$$\left.\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{1+\log y}{1+\log z} \\ \frac{\partial z}{\partial x} &= -\frac{1+\log x}{1+\log z} \end{aligned}\right\} \quad \text{...(2)}$$
Similarly,

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + (1+\log z)\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1+\log z)}\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \quad \text{...(3)}$$

When $x = y = z$

From (2), $\frac{\partial z}{\partial y} = -1, \frac{\partial z}{\partial x} = -1$

From (3), $$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)}(-1)(-1)$$

$$= -\frac{1}{x(\log e + \log x)} = -\frac{1}{x(\log ex)} = -(x \log ex)^{-1}.$$

Example 7. *If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

Sol. $$r^2 = x^2 + y^2 \quad \text{...(1)}$$

Differentiating partially w.r.t. x, we get $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now $u = f(r)$

$\therefore$ $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r) + x\cdot\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r) + \frac{x}{r}f''(r)\cdot\frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial}{\partial x}(u) + uw\frac{\partial}{\partial x}(v) + uv\frac{\partial}{\partial x}(w)\right]$$

$$= \frac{1}{r}f'(r) - \frac{x}{r^2}\cdot\frac{x}{r}f'(r) + \frac{x}{r}\cdot f''(r)\cdot\frac{x}{r} = \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r)$$

$$= \frac{r^2 - x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) = \frac{y^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) \qquad | \text{ Using (1)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3}f'(r) + \frac{y^2}{r^2}f''(r)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2}{r^3}f'(r) + \frac{x^2+y^2}{r^2}f''(r) = \frac{r^2}{r^3}f'(r) + \frac{r^2}{r^2}f''(r) = f''(r) + \frac{1}{r}f'(r).$$

Example 8. *If* $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Sol. Given
$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \qquad \text{...(1)}$$

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1}-x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x}=0$$

or
$$\frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

or
$$\frac{2x}{a^2+u}=V\frac{\partial u}{\partial x}, \text{ where } V=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

or
$$\frac{\partial u}{\partial x}=\frac{2x}{V(a^2+u)}$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{2y}{V(b^2+u)} \text{ and } \frac{\partial u}{\partial z}=\frac{2z}{V(c^2+u)}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{4}{V^2}(V)=\frac{4}{V} \qquad \text{...(2)}$$

Now,
$$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=2\left[\frac{2x^2}{V(a^2+u)}+\frac{2y^2}{V(b^2+u)}+\frac{2z^2}{V(c^2+u)}\right]$$

$$=\frac{4}{V}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]=\frac{4}{V}(1) \qquad \text{[Using (1)]}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2. \qquad \text{[Using (2)]}$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions:
   (*i*) $u=y^x$ (*ii*) $u=\log(x^2+y^2)$
   (*iii*) $u=x^2\sin\dfrac{y}{x}$ (*iv*) $u=\dfrac{x}{y}\tan^{-1}\left(\dfrac{y}{x}\right)$.
2. If $u=x^2+y^2+z^2$, show that $xu_x+yu_y+zu_z=2u$.
3. If $z=\log(x^2+xy+y^2)$, show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2$.
4. If $u=x^2y+y^2z+z^2x$, show that $u_x+u_y+u_z=(x+y+z)^2$.

5. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

6. If $f(x, y) = x^3y - xy^3$, find $\left[\frac{1}{\frac{\partial f}{\partial x}} + \frac{1}{\frac{\partial f}{\partial y}}\right]_{\substack{x=1\\y=2}}$.

7. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$.

8. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, show that $f_x + f_y + f_z = 0$.

9. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(*i*) $u = ax^2 + 2hxy + by^2$ (*ii*) $u = \tan^{-1}\left(\frac{x}{y}\right)$ (*iii*) $u = \log\left(\frac{x^2 + y^2}{xy}\right)$.

10. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$; where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$.

11. If $u = \tan^{-1}\frac{xy}{\sqrt{1 + x^2 + y^2}}$, show that $\frac{\partial^2 u}{\partial x \partial y} = (1 + x^2 + y^2)^{-3/2}$.

12. If $u = e^{xyz}$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$.

13. If $u = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

14. If $z = f(x + ay) + \phi(x - ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

15. Show that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$, where $z = xf(x + y) + yg(x + y)$.

16. Find the value of n so that the equation $V = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

17. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$.

18. If $V = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$.

19. If $V = r^m$, where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m(m + 1)r^{m-2}$.

20. If $u = \log(x^2 + y^2 + z^2)$, show that $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$.

21. If $u = \log\sqrt{x^2 + y^2 + z^2}$, show that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$.

22. If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$, find the value of $\dfrac{\partial^2 u}{\partial x \partial y}$.

23. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, show that $\dfrac{\partial}{\partial y}\left(u \dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(u \dfrac{\partial v}{\partial y}\right)$.

24. If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(i) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 1$ (ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

25. If $V = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $V_{xx} + V_{yy} + V_{zz} = f''(r) + \dfrac{2}{r} f'(r)$.

26. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2} = \left(\dfrac{\partial v}{\partial y}\right)_x \cdot \left(\dfrac{\partial y}{\partial v}\right)_u$.

27. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}$.

[**Hint.** $u = \log\{x^2(x - y) - y^2(x - y)\} = \log(x - y)(x^2 - y^2)$
$= \log(x - y)^2(x + y) = 2\log(x - y) + \log(x + y)$].

28. Find p and q, if $x = \sqrt{a}(\sin u + \cos v)$, $y = \sqrt{a}(\cos u - \sin v)$, $z = 1 + \sin(u - v)$ where p and q mean $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ respectively.

$\left[\textbf{Hint.}\, x^2 + y^2 = 2az, \quad \therefore \quad z = \dfrac{x^2 + y^2}{2a}\right]$.

Answers

1. (i) $y^x \log y, xy^{x-1}$ (ii) $\dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}$ (iii) $2x \sin \dfrac{y}{x} - y \cos \dfrac{y}{x}, x \cos \dfrac{y}{x}$

(iv) $\dfrac{-x}{x^2 + y^2} + \dfrac{1}{y}\tan^{-1}\dfrac{y}{x}, \dfrac{x^2}{y(x^2 + y^2)} - \dfrac{x}{y^2}\tan^{-1}\dfrac{y}{x}$ **6.** $-\dfrac{13}{22}$ **16.** 2, – 3

22. $\dfrac{x^2 - y^2}{x^2 + y^2}$ **28.** $p = \dfrac{x}{a}, q = \dfrac{y}{a}$.

5.10. HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\dfrac{y}{x}\right)$ or $y^n \phi\left(\dfrac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $f(x, y) = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, then

$$(i)\ f(x, y) = \frac{x\left(1 + \dfrac{y}{x}\right)}{\sqrt{x}\left(1 + \sqrt{\dfrac{y}{x}}\right)} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(ii)\ f(x, y) = \frac{y\left(\frac{x}{y} + 1\right)}{\sqrt{y}\left(\sqrt{\frac{x}{y}} + 1\right)} = y^{1/2}\,\phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(iii)\ f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}\,(\sqrt{x} + \sqrt{y})} = t^{1/2} f(x, y)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree (or order) n in the variables x, y, z if $f(x, y, z) = x^n\,\phi\left(\frac{y}{x}, \frac{z}{x}\right)$ or $y^n\,\phi\left(\frac{x}{y}, \frac{z}{y}\right)$ or $z^n\,\phi\left(\frac{x}{z}, \frac{y}{z}\right)$.

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$.

5.11. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function of degree n in x and y then $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$

Since u is a homogeneous function of degree n in x and y, it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \ldots(1)$$

Also
$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Adding (1) and (2), we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nx^n f\left(\dfrac{y}{x}\right) = nu$.

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. Thus, if u is a homogeneous function of degree n in x, y and z, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = nu$.

5.12. IF u IS A HOMOGENEOUS FUNCTION OF DEGREE n IN x AND y, THEN

$$\mathbf{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.}$$

Since u is a homogeneous function of degree n in x and y

$\therefore$ By Euler's Theorem, we have $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = nu$ ...(1)

Differentiating (1) partially w.r.t. x, we have $1 . \dfrac{\partial u}{\partial x} + x \dfrac{\partial^2 u}{\partial x^2} + y . \dfrac{\partial^2 u}{\partial x \partial y} = n \dfrac{\partial u}{\partial x}$ ...(2)

Differentiating (1) partially w.r.t. y, we have $x \dfrac{\partial^2 u}{\partial y \partial x} + 1 . \dfrac{\partial u}{\partial y} + y . \dfrac{\partial^2 u}{\partial y^2} = n \dfrac{\partial u}{\partial y}$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \quad x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y} \quad ...(3)$$

Multiplying (2) by x, (3) by y and adding

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = n \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n . nu \quad \text{[Using (1)]}$$

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n^2 u - nu = n(n-1)u.$$

5.13. DEDUCTIONS FROM EULER'S THEOREM

If $F(u) = V(x, y)$, where V is a homogeneous function in x and y of degree n, then

(1) $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{F(u)}{F'(u)}.$$

(2) $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u)\, [\phi'(u) - 1] \quad \textit{where} \quad \phi(u) = n \frac{F(u)}{F'(u)}.$$

Proof. (1) Since $V = F(u)$ is a homogeneous function of degree n then by Euler's theorem, we get

$$x \frac{\partial}{\partial x} \{F(u)\} + y \frac{\partial}{\partial y} \{F(u)\} = nF(u)$$

or $$xF'(u) \frac{\partial u}{\partial x} + yF'(u) \frac{\partial u}{\partial y} = nF(u)$$

or $$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{F(u)}{F'(u)}}. \quad ...(i)$$

(2) Let $n \dfrac{F(u)}{F'(u)} = \phi(u)$ then from (*i*),

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \phi(u) \quad ...(ii)$$

Differentiating (*ii*) partially w.r.t. x, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \phi'(u) \frac{\partial u}{\partial x}$$

$$\Rightarrow \qquad x\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 u}{\partial x \partial y} = \{\phi'(u) - 1\}\,\frac{\partial u}{\partial x} \qquad ...(iii)$$

Similarly, differentiating (ii) partially w.r.t. y, we get

$$y\,\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial y \partial x} = \{\phi'(u) - 1\}\,\frac{\partial u}{\partial y} \qquad ...(iv)$$

Multiplying (iii) by x, (iv) by y and adding, we get

$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2xy\,\frac{\partial^2 u}{\partial x \partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = \{\phi'(u) - 1\}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \{\phi'(u) - 1\}\,\phi(u) \qquad | \text{ From } (ii)$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Euler's theorem for the functions:*

(i) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ $\qquad$ (ii) $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$.

Sol. (i) $\quad u(x, y) = (x^{1/2} + y^{1/2})(x^n + y^n) \qquad ...(1)$

$$u(tx, ty) = t^{1/2}\,(x^{1/2} + y^{1/2})\,t^n\,(x^n + y^n) = t^{n+1/2}\,u(x, y)$$

$\Rightarrow$ u is a homogeneous function of degree $(n + \frac{1}{2})$ in x and y

$\therefore$ By Euler's theorem, we should have $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = \left(n + \dfrac{1}{2}\right)u \qquad ...(2)$

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{2}\,x^{-1/2}\,(x^n + y^n) + nx^{n-1}(x^{1/2} + y^{1/2})$$

$$x\,\frac{\partial u}{\partial x} = \frac{1}{2}\,x^{1/2}\,(x^n + y^n) + nx^n(x^{1/2} + y^{1/2})$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\,y^{-1/2}\,(x^n + y^n) + ny^{n-1}\,(x^{1/2} + y^{1/2})$$

$$y\,\frac{\partial u}{\partial y} = \frac{1}{2}\,y^{1/2}(x^n + y^n) + ny^n(x^{1/2} + y^{1/2})$$

$$\therefore \qquad x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = \frac{1}{2}\,(x^{1/2} + y^{1/2})(x^n + y^n) + n(x^n + y^n)(x^{1/2} + y^{1/2})$$

$= \frac{1}{2}\,u + nu = (n + \frac{1}{2})u$ which is the same as (2). Hence the verification.

(ii) $\qquad u(x, y) = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x} \qquad ...(1)$

$$u(tx, ty) = \sin^{-1}\frac{tx}{ty} + \tan^{-1}\frac{ty}{tx} = t^{\circ}\left(\sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}\right) = t^{\circ}u(x, y)$$

$\Rightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, we should have $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = 0 \times u = 0 \qquad ...(2)$

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \dfrac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1 + \dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{\sqrt{x^2 + y^2}}$$

$$x\,\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$y\,\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$\therefore \quad x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = 0$ which is the same as (2). Hence the verification.

Example 2. *(i) If* $u = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(ii) Show that $x\,\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$ *where* $u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$.

Sol. *(i)*
$$u\,(x, y, z) = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$$

$\therefore$
$$u(tx, ty, tz) = \cos\left\{\frac{t^2(xy + yz + zx)}{t^2(x^2 + y^2 + z^2)}\right\}$$

$$= t^\circ \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right) = t^\circ u\,(x, y, z)$$

Hence u is a homogeneous function in x, y and z of degree 0.
Hence by Euler's theorem,

$$x\,\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\,\frac{\partial u}{\partial z} = 0\,.\,u = 0$$

(ii) u is not a homogeneous function.

Let,
$$\sin u = \frac{x^3 + y^3 + z^3}{ax + by + cz} = f(x, y, z) \qquad \text{...(1)} \quad |\text{ say}$$

Now,
$$f(tx, ty, tz) = \frac{t^3(x^3 + y^3 + z^3)}{t(ax + by + cz)} = t^2 f(x, y, z)$$

Hence $f(x, y, z)$ is a homogeneous function of degree 2.
$\therefore$ By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 2f$$

$$\Rightarrow \quad x\,\frac{\partial}{\partial x}\,(\sin u) + y\,\frac{\partial}{\partial y}\,(\sin u) + z\,\frac{\partial}{\partial z}(\sin u) = 2\sin u$$

$$\Rightarrow \quad x\cos u\,\frac{\partial u}{\partial x} + y\cos u\,\frac{\partial u}{\partial y} + z\cos u\,\frac{\partial u}{\partial z} = 2\sin u$$

$$\Rightarrow \quad x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} + z\,\frac{\partial u}{\partial z} = 2\tan u.$$

Example 3. *If* $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, *prove that*

(*i*) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u$

(*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2 \cos 3u \sin u.$

Sol. (*i*) Here u is not a homogeneous function.

Let $$\tan u = \frac{x^3 + y^3}{x - y} = f(x, y) \quad \text{...(1)} \quad | \text{ say}$$

Now, $$f(tx, ty) = \frac{t^3(x^3 + y^3)}{t(x - y)} = t^2 f(x, y)$$

Hence $f(x, y) = \tan u$ is a homogeneous function of degree 2.

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$$

or $$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \sin u}{\cos u} . \cos^2 u = 2 \sin u \cos u = \sin 2u. \quad \text{...(2)}$$

(*ii*) Here $\phi(u) = \sin 2u$

∴ $\phi'(u) = 2 \cos 2u$

By deduction from Euler's theorem, we know that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u) \{\phi'(u) - 1\}$$
$$= \sin 2u (2 \cos 2u - 1) = \sin 4u - \sin 2u$$
$$= 2 \cos 3u \sin u.$$

Example 4. *If* $u = \sin^{-1} \left(\dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}} \right)$, *show that*

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$

Sol. Here u is not a homogeneous function. But, we have

$$\sin u = \frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}} = f(x, y, z) \quad | \text{ say}$$

$$f(tx, ty, tz) = \frac{t(x + 2y + 3z)}{t^4 \sqrt{x^8 + y^8 + z^8}} = t^{-3} f(x, y, z)$$

∴ $\sin u$ is a homogeneous function of degree -3 in x, y, z.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) + z \frac{\partial}{\partial z} (\sin u) = -3 \sin u$$

or $$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} + 3 \sin u = 0$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$

Example 5. *If* $u = \log_e \left(\frac{x^4 + y^4}{x + y} \right)$, *show that* $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.$

Sol. Here u is not a homogeneous function

$$u = \log_e \left(\frac{x^4 + y^4}{x + y} \right)$$

$\Rightarrow$ $$e^u = \frac{x^4 + y^4}{x + y}$$

which is a homogeneous function of degree 3 in x, y.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = 3e^u$$

or $$xe^u \frac{\partial u}{\partial x} + y\, e^u \frac{\partial u}{\partial y} = 3e^u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.$$

Example 6. *If* $u = \sin^{-1} \frac{x + y}{\sqrt{x} + \sqrt{y}}$, *prove that*

(*i*) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$ (*ii*) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4 \cos^3 u}.$

Sol. (*i*) $u = \sin^{-1} \frac{x + y}{\sqrt{x} + \sqrt{y}}$ is not a homogeneous function but

$\sin u = \frac{x + y}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = \frac{1}{2} \sin u$$

or $$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \quad \ldots(1)$$

(*ii*) Here $$\phi(u) = \frac{1}{2} \tan u$$

$\therefore$ $$\phi'(u) = \frac{1}{2} \sec^2 u$$

By deduction from Euler's theorem, we know that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u) \{\phi'(u) - 1\}$$

$$= \frac{1}{2} \tan u \left(\frac{1}{2} \sec^2 u - 1\right) = \frac{1}{2} \frac{\sin u}{\cos u} \left(\frac{1}{2 \cos^2 u} - 1\right)$$

$$= \frac{1}{2} \frac{\sin u}{\cos u} \left\{\frac{-(2 \cos^2 u - 1)}{2 \cos^2 u}\right\} = -\frac{\sin u \cos 2u}{4 \cos^3 u}.$$

Example 7. *If* $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, *show that*

(*i*) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x\, \phi\left(\frac{y}{x}\right)$ (*ii*) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$

Sol. Let $v = x\, \phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right) = x^0\, \psi\left(\frac{y}{x}\right)$ so that,

$$u = v + w \qquad \text{...(1)}$$

(*i*) Since v is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v \qquad \text{...(2)}$$

Since w is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 0 \qquad \text{...(3)}$$

Adding equations (2) and (3), we get

$$x \frac{\partial}{\partial x} (v + w) + y \frac{\partial}{\partial y} (v + w) = v$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x\, \phi\left(\frac{y}{x}\right).$$

(*ii*) Since v is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n\,(n-1)\, v = 1\,(1-1)\, v = 0 \qquad \text{...(4)}$$

Also since w is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = n(n-1)\, w = 0\,(0-1)\, w = 0 \qquad \text{...(5)}$$

Adding (4) and (5), we get

$$x^2 \frac{\partial^2}{\partial x^2} (v + w) + 2xy \frac{\partial^2}{\partial x \partial y} (v + w) + y^2 \frac{\partial^2}{\partial y^2} (v + w) = 0$$

$$\Rightarrow \quad x^2 \left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy \left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2 \left(\frac{\partial^2 u}{\partial y^2}\right) = 0.$$

TEST YOUR KNOWLEDGE

1. Verify Euler's theorem for the following functions:

(*i*) $f(x, y) = ax^2 + 2hxy + by^2$ (*ii*) $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$

(*iii*) $f(x, y) = \dfrac{x^2(x^2 - y^2)^3}{(x^2 + y^2)^3}$ (*iv*) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

(*v*) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$. (*vi*) $u = x^2 \log\left(\dfrac{y}{x}\right)$.

2. (*i*) If $u = f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (*ii*) If $u = xf\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

3. If $V = \dfrac{x^3y^3}{x^3 + y^3}$, prove that $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} = 3V$.

4. If $u(x, y) = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

5. If $f(x, y) = \sqrt{x^2 - y^2}\, \sin^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f(x, y)$.

6. If $f(x, y) = \sqrt{y^2 - x^2}\, \sin^{-1}\dfrac{x}{y} + \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$, prove that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} - f(x, y) = 0$.

7. If $f(x, y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$, prove that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + 2f(x, y) = 0$.

8. (*i*) If $z = xy\, f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$.

(*ii*) If $x = e^u \tan v$, $y = e^u \sec v$; find the value of $\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right).\left(x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}\right)$.

9. (*i*) If $u = \cos^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

(*ii*) If $u = \log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

10. (*i*) If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

(*ii*) If $u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$.

Also evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

11. (*a*) If $u = \sin^{-1}\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, prove that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

(*b*) If $\sin u = \dfrac{x^2y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$.

12. (*a*) If $u = \log\left(\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$.

(*b*) If $u = \dfrac{x^2y^2z^2}{x^2 + y^2 + z^2} + \cos\dfrac{xy + yz}{x^2 + y^2 + z^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = \dfrac{4x^2y^2z^2}{x^2 + y^2 + z^2}$.

13. If $u = \dfrac{x^2y^2}{x + y}$, prove that

(*i*) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = 2\dfrac{\partial u}{\partial x}$ (*ii*) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = 2\dfrac{\partial u}{\partial y}$.

14. Given $z = x^n f_1\left(\dfrac{y}{x}\right) + y^{-n} f_2\left(\dfrac{x}{y}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.

15. If $u = (x^2 + y^2)^{1/3}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2u}{9}$.

16. If $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x, y of degree m and n respectively and $u = f(x, y) + \phi(x, y)$, prove that

$$f(x, y) = \frac{1}{m(m - n)}\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) - \frac{n - 1}{m(m - n)}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right).$$

17. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

18. (*i*) If $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

(*ii*) If $u = x \sin^{-1}\left(\dfrac{x}{y}\right) + y \sin^{-1}\left(\dfrac{y}{x}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

19. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13 + \tan^2 u)$.

20. If $V = \log_e \sin\left\{\dfrac{\pi(2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}}\right\}$, prove that when $x = 0, y = 1, z = 2$;

$x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} + z\dfrac{\partial V}{\partial z} = \dfrac{\pi}{12}$.

Answers

8. (*ii*) 0 **10.** (*ii*) $-2\cot u\,(2\operatorname{cosec}^2 u + 1)$ **18.** (*i*) $2u$ (*ii*) 0.

5.14. COMPOSITE FUNCTIONS

(*i*) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then u is called a composite function of (the **single variable**) t and we can find $\dfrac{du}{dt}$.

(*ii*) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is called a composite function of (**two variables**) u and v so that we can find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

5.15. DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.}$$

Proof. Here $\qquad u = f(x, y) \qquad \ldots(1)$

Let δt be an increment in t and δx, δy, δu the corresponding increments in x, y and u respectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \qquad \ldots(2)$$

Subtracting (1) from (2), we get

$$\begin{aligned}\delta u &= f(x + \delta x, y + \delta y) - f(x, y)\\ &= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)\\ \frac{\delta u}{\delta t} &= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}\\ &= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t} \qquad \ldots(3)\end{aligned}$$

As $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta u}{\delta t} = \frac{du}{dt},\ \underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{\delta t} = \frac{dx}{dt},\ \underset{\delta t \to 0}{\text{Lt}} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$

and

$$\underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

$\therefore$ From (1), $\displaystyle\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$\dfrac{du}{dt}$ is called the **total derivative** of u to distinguish it from the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

Cor. 1. If $u = f(x, y, z)$ and x, y, z are function of t, then u is a composite function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

Cor. 2. If $z = f(x, y)$ and x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u};\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Cor. 3. If $u = f(x, y)$ where $y = \phi(x)$ then since $x = \psi(x)$, u is a composite function of x.

$$\therefore \qquad \frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

Cor. 4. If we are given an implicit function $f(x, y) = c$, then $u = f(x, y)$ where $u = c$

Using Cor. 3, we have $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$

But $\dfrac{du}{dx} = 0 \quad \therefore \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$ or $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx} = 0$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Cor. 5. If $f(x, y) = c$, then by Cor. 4, we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{f_y \dfrac{d}{dx}(f_x) - f_x \dfrac{d}{dx}(f_y)}{f_y^{\,2}}\right] = -\frac{f_y\left[\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_x}{\partial y} \cdot \dfrac{dy}{dx}\right] - f_x\left[\dfrac{\partial f_y}{\partial x} + \dfrac{\partial f_y}{\partial y} \cdot \dfrac{dy}{dx}\right]}{f_y^{\,2}}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx} \cdot \dfrac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy} \cdot \dfrac{f_x}{f_y}\right]}{f_y^{\,2}} = -\left[\frac{f_{xx}f_y^{\,2} - f_xf_yf_{xy} - f_xf_yf_{xy} + f_{yy}f_x^{\,2}}{f_y^{\,3}}\right]$$

Hence $\mathbf{\dfrac{d^2y}{dx^2} = -\left(\dfrac{f_{xx}f_y^{\,2} - 2f_xf_yf_{xy} + f_{yy}f_x^{\,2}}{f_y^{\,3}}\right)}$.

ILLUSTRATIVE EXAMPLES

Example 1. *If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

Sol. The given equations define u as a composite function of t.

$$\therefore \qquad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1 - (x - y)^2}} \cdot 3 + \frac{1}{\sqrt{1 - (x - y)^2}} (-1) \cdot 12t^2$$

$$= \frac{3(1 - 4t^2)}{\sqrt{1 - (3t - 4t^3)^2}} = \frac{3(1 - 4t^2)}{\sqrt{1 - 9t^2 + 24t^4 - 16t^6}}$$

$$= \frac{3(1 - 4t^2)}{\sqrt{(1 - t^2)(1 - 8t^2 + 16t^4)}} = \frac{3(1 - 4t^2)}{\sqrt{(1 - t^2)(1 - 4t^2)^2}} = \frac{3}{\sqrt{1 - t^2}}.$$

Example 2. (*i*) *If* $u = x \log (xy)$ *where* $x^3 + y^3 + 3xy = 1$, *find* $\frac{du}{dx}$

(*ii*) *If* $u = f(r, s)$ *and* $r = x + y$, $s = x - y$; *show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Sol. (*i*) $$u = x \log xy \qquad ...(1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = x\left(\frac{1}{xy} . y\right) + \log (xy) = 1 + \log xy$$

and $$\frac{\partial u}{\partial y} = x\left(\frac{1}{xy} . x\right) = \frac{x}{y}$$

Also, $x^3 + y^3 + 3xy = 1$ ...(2)

Differentiating (2), we get

$$3x^2 + 3y^2 \frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x^2 + y}{x + y^2}\right) \qquad ...(3)$$

we know that $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 1 + \log (xy) + \frac{x}{y}\left\{-\left(\frac{x^2 + y}{x + y^2}\right)\right\}$$

$$= 1 + \log (xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$$

(*ii*) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} . \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad ...(1) \qquad \left| \because \frac{\partial r}{\partial x} = \frac{\partial s}{\partial x} = 1 \right.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} . \frac{\partial s}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \qquad ...(2) \qquad \left| \because \frac{\partial r}{\partial y} = 1, \frac{\partial s}{\partial y} = -1 \right.$$

Adding (1) and (2), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}.$$

Example 3. *If z is a function of x and y, where* $x = e^u + e^{-v}$ *and* $y = e^{-u} - e^v$, *show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Sol. Here z is a composite function of u and v.

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} . e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

Subtracting, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 4. *If* $u = f(y - z, z - x, x - y)$, *prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol. Here, $u = f(X, Y, Z)$, where $X = y - z$, $Y = z - x$, $Z = x - y$

$\therefore$ u is a composite function of x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding, we get $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Example 5. *If* $f(x, y) = 0$, $\phi(y, z) = 0$, *show that* $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$.

Sol. $$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} \qquad ...(1)$$

$$f(x, y) = 0 \text{ gives } \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\phi(y, z) = 0 \text{ gives } \frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

$\therefore$ From (1), $$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}} \quad \Rightarrow \quad \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}.$$

Example 6. *If* $\phi(x, y, z) = 0$, *show that* $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$.

Sol. The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

Similarly, $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$$

Multiplying, we get the desired result.

Example 7. *(i) If* $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, *show that* $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.

(ii) If $\phi(cx - az, cy - bz) = 0$, *show that* $a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = c$.

Sol. *(i)* Let $$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \qquad ...(1)$$

and $$w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z} \qquad ...(2)$$

so that $$u = u(v, w) \qquad ...(3)$$

Now, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial w}\left(\frac{-1}{x^2}\right) \qquad | \text{ Using (1) and (2)}$$

$$\Rightarrow \quad x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \qquad ...(4)$$

Also, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \qquad | \text{ Using (1) and (2)}$$

$$\Rightarrow \quad y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \qquad ...(5)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right)$$

$$\Rightarrow \quad z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \qquad ...(6)$$

Adding (4), (5) and (6), we get

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

(ii) Let $v = cx - az$ and $w = cy - bz$ then we have

$$\phi(v, w) = 0 \qquad ...(1)$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$0 = \frac{\partial \phi}{\partial v}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial w}\left(-b\frac{\partial z}{\partial x}\right)$$

$$0 = c\frac{\partial \phi}{\partial v} + \frac{\partial z}{\partial x}\left(-a\frac{\partial \phi}{\partial v} - b\frac{\partial \phi}{\partial w}\right)$$

$$\Rightarrow \quad a\frac{\partial z}{\partial x} = \frac{ac\dfrac{\partial \phi}{\partial v}}{a\dfrac{\partial \phi}{\partial v} + b\dfrac{\partial \phi}{\partial w}} \qquad ...(2)$$

Also, $$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$0 = \frac{\partial \phi}{\partial v}\left(-a\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial w}\left(c - b\frac{\partial z}{\partial y}\right)$$

$$0 = c\frac{\partial \phi}{\partial w} + \frac{\partial z}{\partial y}\left(-a\frac{\partial \phi}{\partial v} - b\frac{\partial \phi}{\partial w}\right)$$

$$\Rightarrow \qquad b\,\frac{\partial z}{\partial y} = \frac{bc\,\dfrac{\partial \phi}{\partial w}}{a\dfrac{\partial \phi}{\partial v} + b\dfrac{\partial \phi}{\partial w}} \qquad \ldots(3)$$

Adding (2) and (3), we get

$$a\,\frac{\partial z}{\partial x} + b\,\frac{\partial z}{\partial y} = \frac{c\left(a\dfrac{\partial \phi}{\partial v} + b\dfrac{\partial \phi}{\partial w}\right)}{\left(a\dfrac{\partial \phi}{\partial v} + b\dfrac{\partial \phi}{\partial w}\right)} = c$$

Example 8. *Prove that* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$, *where*

$$x = u\cos\alpha - v\sin\alpha,\; y = u\sin\alpha + v\cos\alpha$$

Sol. Here z is a composite function of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \cos\alpha\,\frac{\partial z}{\partial x} + \sin\alpha\,\frac{\partial z}{\partial y}$$

$$\Rightarrow \qquad \frac{\partial}{\partial u} \equiv \cos\alpha\,\frac{\partial}{\partial x} + \sin\alpha\,\frac{\partial}{\partial y} \qquad \ldots(1)$$

Also

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -\sin\alpha\,\frac{\partial z}{\partial x} + \cos\alpha\,\frac{\partial z}{\partial y}$$

$$\Rightarrow \qquad \frac{\partial}{\partial v} \equiv -\sin\alpha\,\frac{\partial}{\partial x} + \cos\alpha\,\frac{\partial}{\partial y} \qquad \ldots(2)$$

Now we shall make use of the equivalence of operators as given by (1) and (2).

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha\,\frac{\partial}{\partial x} + \sin\alpha\,\frac{\partial}{\partial y}\right)\left(\cos\alpha\,\frac{\partial z}{\partial x} + \sin\alpha\,\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha\,\frac{\partial^2 z}{\partial x^2} + \cos\alpha\sin\alpha\,\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\cos\alpha\,\frac{\partial^2 z}{\partial y\partial x} + \sin^2\alpha\,\frac{\partial^2 z}{\partial y^2}$$

$$= \cos^2\alpha\,\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\,\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\,\frac{\partial^2 z}{\partial y^2} \qquad \ldots(3)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha\,\frac{\partial}{\partial x} + \cos\alpha\,\frac{\partial}{\partial y}\right)\left(-\sin\alpha\,\frac{\partial z}{\partial x} + \cos\alpha\,\frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha\,\frac{\partial^2 z}{\partial x^2} - \sin\alpha\cos\alpha\,\frac{\partial^2 z}{\partial x\partial y} - \cos\alpha\sin\alpha\,\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\,\frac{\partial^2 z}{\partial y^2}$$

$$= \sin^2\alpha\,\frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha\,\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\,\frac{\partial^2 z}{\partial y^2} \qquad \ldots(4)$$

Adding (3) and (4), $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$.

TEST YOUR KNOWLEDGE

1. If $u = x^3 + y^3$, where $x = a \cos t$, $y = b \sin t$, find $\dfrac{du}{dt}$ and verify the result.

2. If $z = u^2 + v^2$, $u = r \cos \theta$, $v = r \sin \theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

3. (*i*) If $x^3 + y^3 - 3axy = 0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ using partial derivative method.

(*ii*) If $z = x^2y$ and $x^2 + xy + y^2 = 1$, show that $\dfrac{dz}{dx} = 2\,xy - \dfrac{x^2(2x + y)}{x + 2y}$.

4. If the curves $f(x, y) = 0$ and $\phi\,(x, y) = 0$ touch each other, show that at the point of contact

$$\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial x} = 0.$$

5. If $x = u + v$, $y = uv$ and z is a function of x, y; show that $u\,\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y}$.

6. If $u = f(r, s)$, $r = x + at$, $s = y + bt$, show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$.

7. If $w = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$, show that: $y\dfrac{\partial w}{\partial u} + x\dfrac{\partial w}{\partial v} = e^{2u}\,\dfrac{\partial w}{\partial y}$.

8. If $x = e^r \cos \theta$, $y = e^r \sin \theta$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial^2 u}{\partial \theta^2}\right)$.

9. If $v = f(r)$; $r^2 = x^2 + y^2 + z^2$, prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial v}{\partial r}$.

10. (*a*) If $u = f(r, s, t)$ and $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$, $t = \dfrac{z}{x}$, prove that $x\,\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

(*b*) If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

(*c*) If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x} + \dfrac{1}{3}\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\dfrac{\partial u}{\partial z} = 0$.

(*d*) If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\,\dfrac{\partial u}{\partial x} + (x^2 - yz)\,\dfrac{\partial u}{\partial y} + (z^2 - xy)\,\dfrac{\partial u}{\partial z} = 0$.

11. If $z = xyf\left(\dfrac{y}{x}\right)$ and z is a constant, show that $\dfrac{f'\left(\dfrac{y}{x}\right)}{f\left(\dfrac{y}{x}\right)} = \dfrac{x\left[y + x\dfrac{dy}{dx}\right]}{y\left[y - x\dfrac{dy}{dx}\right]}$.

12. If z is a function of x and y, and u and v be two other variables such that $u = lx + my$, $v = ly - mx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

Answers

1. $-3a^3 \cos^2 t \sin t + 3b^3 \sin^2 t \cos t$ **2.** $2r, 0$ **3.** (i) $\dfrac{x^2 - ay}{ax - y^2}, \dfrac{2a^3xy}{(ax - y^2)^3}$.

EXPANSION OF A FUNCTION OF TWO VARIABLES

5.16. TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

We know that by Taylor's theorem for a function $f(x)$ of single variable x,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

Now let $f(x, y)$ be a function of two independent variables x and y. **If y is kept constant,** then by Taylor's theorem for a function of a single variable x, we have

$$f(x + h, \mathbf{y + k}) = f(x, \mathbf{y + k}) + h \frac{\partial}{\partial x} f(x, \mathbf{y + k}) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x, \mathbf{y + k})$$
$$+ \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} f(x, \mathbf{y + k}) + \ldots\ldots \quad \ldots(1)$$

Now **keeping x constant** and applying Taylor's theorem for a function of a single variable y, we have

$$f(\mathbf{x}, y + k) = f(\mathbf{x}, y) + k \frac{\partial}{\partial y} f(\mathbf{x}, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(\mathbf{x}, y) + \frac{k^3}{3!} \frac{\partial^3}{\partial y^3} f(\mathbf{x}, y) + \ldots\ldots \quad \ldots(2)$$

Using (2), we can write (1) as

$$f(x + h, y + k) = \left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!} \frac{\partial^3}{\partial y^3} f(x, y) + \ldots\right]$$
$$+ h\frac{\partial}{\partial x}\left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \ldots\right]$$
$$+ \frac{h^2}{2!} \frac{\partial^2}{\partial x^2}\left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \ldots\right] + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} [f(x, y) + \ldots] + \ldots\ldots$$
$$= \left[f(x, y) + k\frac{\partial f}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f}{\partial y^3} + \ldots\right] + \left[h\frac{\partial f}{\partial x} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{hk^2}{2!} \frac{\partial^3 f}{\partial x \partial y^2} + \ldots\right]$$
$$+ \left[\frac{h^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{h^2 k}{2!} \frac{\partial^3 f}{\partial x^2 \partial y} + \ldots\right] + \frac{h^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!}\frac{\partial^3 f}{\partial x^2 \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right) + \ldots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \ldots$$

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots..$$

Hence,

$$\mathbf{f(x+h,\ y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + \frac{1}{3!}(h^3 f_{xxx} + 3h^2 kf_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) + \ldots\ldots}$$

Cor. 1. Putting $x = a$ and $y = b$, we have

$$f(a+h, b+k) = f(a, b) + \{hf_x(a, b) + kf_y(a, b)\} + \frac{1}{2!}\{h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\} + \frac{1}{3!}\{h^3 f_{xxx}(a, b) + 3h^2 kf_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)\} + \ldots\ldots$$

Cor. 2. In cor. 1, put $h = x - a$ and $k = y - b$, we have

$$f(x, y) = f(a, b) + \{(x-a)f_x(a, b) + (y-b)f_y(a, b)\} + \frac{1}{2!}\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\} + \ldots\ldots.$$

Cor. 3. Putting $a = 0$ and $b = 0$ in cor. 2, we have

$$f(x, y) = f(0, 0) + \{(xf_x(0, 0) + yf_y(0, 0)\} + \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\} + \ldots..$$

This is called **Maclaurin's theorem** for two variables.

Note. Cor. 3 is used to expand $f(x, y)$ in powers of x and y [or to expand $f(x, y)$ in the neighbourhood of origin (0, 0)]. Cor. 2 is used to expand $f(x, y)$ in the neighbourhood of (a, b).

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.*

Sol. Here,

$f(x, y) = e^x \sin y;$	$f(0, 0) = 0$
$f_x(x, y) = e^x \sin y,$	$f_x(0, 0) = 0$
$f_y(x, y) = e^x \cos y,$	$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \sin y,$	$f_{xx}(0, 0) = 0$

$$f_{xy}(x, y) = e^x \cos y, \qquad f_{xy}(0, 0) = 1$$
$$f_{yy}(x, y) = -e^x \sin y, \qquad f_{yy}(0, 0) = 0$$
$$f_{xxx}(x, y) = e^x \sin y, \qquad f_{xxx}(0, 0) = 0$$
$$f_{xxy}(x, y) = e^x \cos y, \qquad f_{xxy}(0, 0) = 1$$
$$f_{xyy}(x, y) = -e^x \sin y, \qquad f_{xyy}(0, 0) = 0$$
$$f_{yyy}(x, y) = -e^x \cos y, \qquad f_{yyy}(0, 0) = -1$$

....

$$\therefore \quad f(x, y) = f(0, 0) + [xf_x(0, 0) + y\, f_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$
$$+ \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2y\, f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + ...$$

$$\Rightarrow \quad e^x \sin y = 0 + [x.0 + y.1] + \frac{1}{2!}[x^2.0 + 2xy.1 + y^2.0]$$
$$+ \frac{1}{3!}[x^3.0 + 3x^2y.1 + 3xy^2 + 0 + y^3(-1)] + ...$$
$$= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + ...$$

Example 2. *Expand* $\tan^{-1}\dfrac{y}{x}$ *in the neighbourhood of (1, 1) upto and inclusive of second degree terms. Hence compute f(1.1, 0.9) approximately.*

Sol. Here $\quad f(x, y) = \tan^{-1}\dfrac{y}{x}, \qquad f(1, 1) = \tan^{-1} 1 = \dfrac{\pi}{4}$

$$f_x(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}, \qquad f_x(1, 1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}, \qquad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = -y(-1)(x^2+y^2)^{-2}.\, 2x = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = x(-1)(x^2+y^2)^{-2}.\, 2y = -\frac{2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1, 1) = -\frac{1}{2}$$

Similarly,

$$f_{xxx}(1, 1) = -\frac{1}{2}, \quad f_{xxy}(1, 1) = -\frac{1}{2}, \quad f_{xyy}(1, 1) = \frac{1}{2} \quad \text{and} \quad f_{yyy}(1, 1) = \frac{1}{2}$$

We know that,

$$f(x, y) = f(1, 1) + [(x-1)\, f_x(1, 1) + (y-1)\, f_y(1, 1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1, 1)$$
$$+ 2(x-1)(y-1)\, f_{xy}(1, 1) + (y-1)^2 f_{yy}(1, 1)]$$

$$+ \frac{1}{3!}[(x-1)^3 f_{xxx}(1, 1) + 3(x-1)^2 (y-1) f_{xxy}(1, 1)$$

$$+ 3(x-1)(y-1)^2 f_{xyy}(1, 1) + (y-1)^3 f_{yyy}(1, 1)] + \ldots\ldots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left[(x-1).\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2 . \frac{1}{2} + 2(x-1)(y-1).0 + (y-1)^2 . (-\tfrac{1}{2})\right]$$

$$- \frac{1}{12}[(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3] + \ldots$$

Now, $$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(.1) + \frac{1}{2}(-.1) + \frac{1}{4}(.1)^2 - \frac{1}{4}(-.1)^2$$

$$- \frac{1}{12}[(.1)^3 + 3(.1)^2(-.1) - 3(.1)(-.1)^2 - (-.1)^3] + \ldots$$

$= 0.6857$ approximately.

Example 3. *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's Theorem.*

Sol. Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ is given by

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b)$$

$$+ 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a, b)$$

$$+ 3(x-a)^2(y-b)f_{xxy}(a, b) + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \ldots. \quad \ldots(1)$$

Here $f(x, y) = x^2y + 3y - 2$, $a = 1$, $b = -2$

$f(1, -2) = -10$

$f_x = 2xy,$	$f_x(1, -2) = -4;$	$f_y = x^2 + 3,$	$f_y(1, -2) = 4$
$f_{xx} = 2y,$	$f_{xx}(1, -2) = -4;$	$f_{xy} = 2x,$	$f_{xy}(1, -2) = 2$
$f_{yy} = 0,$	$f_{yy}(1, -2) = 0;$	$f_{xxx} = 0,$	$f_{xxx}(1, -2) = 0$
$f_{xxy} = 2,$	$f_{xxy}(1, -2) = 2;$	$f_{xyy} = 0,$	$f_{xyy}(1, -2) = 0$
$f_{yyy} = 0,$	$f_{yyy}(1, -2) = 0$		

All higher order partial derivatives vanish.

$\therefore$ From (1), we have

$$x^2y + 3y - 1 = f(x, y)$$

$$= -10 + [(x-1)(-4) + (y+2)(4)] + \tfrac{1}{2}[(x-1)^2(-4)$$

$$+ 2(x-1)(y+2)(2) + (y+2)^2(0)] + \tfrac{1}{6}[(x-1)^3(0)$$

$$+ 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example 4. *Expand $\dfrac{(x+h)(y+k)}{x+h+y+k}$ in powers of h, k upto and inclusive of the second degree terms.*

Sol. Here $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

Putting $h = k = 0$, we have $f(x, y) = \dfrac{xy}{x+y}$

$$f_x = \frac{(x+y).y - xy.1}{(x+y)^2} = \frac{y^2}{(x+y)^2}; \qquad f_y = \frac{x^2}{(x+y)^2}, \text{ by symmetry}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}, f_{yy} = -\frac{2x^2}{(x+y)^3}; \qquad f_{xy} = \frac{(x+y)^2 . 2x - x^2 . 2(x+y)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$$

$$\therefore \quad \frac{(x+h)(y+k)}{x+h+y+k} = f(x+h, y+k) = f(x, y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \ldots.$$

$$= \frac{xy}{x+y} + \left[h . \frac{y^2}{(x+y)^2} + k . \frac{x^2}{(x+y)^2}\right]$$

$$+ \frac{1}{2}\left[h^2 . \frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2 . \frac{-2x^2}{(x+y)^3}\right] + \ldots..$$

TEST YOUR KNOWLEDGE

1. Expand $e^{ax} \sin by$ in powers of x and y as far as the terms of third degree.
2. Expand $e^x \cos y$ in powers of x and y as far as the terms of third degree.
3. Expand $e^x \cos y$ at $\left(1, \dfrac{\pi}{4}\right)$.
4. Expand e^{xy} at (1, 1).
5. Expand $(1 + x + y^2)^{1/2}$ at (1, 0).
6. Expand $\sin xy$ in powers of $(x - 1)$ and $(y - \pi/2)$ up to second degree terms.
7. Expand x^y in powers of $(x - 1)$ and $(y - 1)$ up to the third degree terms and hence evaluate $(1.1)^{1.02}$.
8. Expand $\sin (x + h)(y + k)$ by Taylor's Theorem.

Answers

1. $by + abxy + \dfrac{1}{3!}(3a^2 bx^2y - b^3y^3) + \ldots..$
2. $1 + x + \dfrac{1}{2!}(x^2 - y^2) + \dfrac{1}{3!}(x^3 - 3xy^2) + \ldots..$
3. $\dfrac{e}{\sqrt{2}}\left\{1 + (x-1) - \left(y - \dfrac{\pi}{4}\right) + \dfrac{(x-1)^2}{2} - (x-1)\left(y - \dfrac{\pi}{4}\right) - \dfrac{1}{2}\left(y - \dfrac{\pi}{4}\right)^2 + \ldots\right\}$
4. $e\left\{1 + (x-1) + (y-1) + \dfrac{1}{2!}\left((x-1)^2 + 4(x-1)(y-1) + (y-1)^2\right) + \ldots\right\}$
5. $\sqrt{2}\left\{1 + \dfrac{x-1}{4} - \dfrac{(x-1)^2}{32} + \dfrac{y^2}{4} + \ldots\right\}$
6. $1 - \dfrac{\pi^2}{8}(x-1)^2 - \dfrac{\pi}{2}(x-1)\left(y - \dfrac{\pi}{2}\right) - \dfrac{1}{2}\left(y - \dfrac{\pi}{2}\right)^2$
7. $1 + (x-1) + (x-1)(y-1) + \dfrac{1}{2}(x-1)^2(y-1) + \ldots..$; 1.1021.
8. $\sin xy + (hy + kx)\cos xy + hk \cos xy - \dfrac{1}{2}(hy + kx)^2 \sin xy + \ldots..$